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Improved correlation dimension estimates through change of variable

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To evaluate the correlation dimension of chaotic regimes of a CO_2 laser with modulated losses, attractor reconstruction using the method of delays is performed using the logarithm of the intensity rather than the intensity itself. Improved convergence with respect to the embedding dimension and better reliability are observed.

1. Introduction

The measure of quantities like dimensions, entropies and Lyapunov exponents has by now become standard to analyze and characterize the chaotic dynamics which can be observed in a variety of experimental systems [1]. Most of the time, they are evaluated by reconstructing from a single time series the underlying strange attractor in $\mathbb{R}^n$ using the method of time delays proposed by Takens and Packard et al. [2]. Indeed, in the limit of an infinite, noise-free time series, the reconstructed attractor is shown to be diffeomorphic to the original one and may therefore be used to compute the above-mentioned quantities, which are left invariant by diffeomorphisms. Nevertheless the experimentalist is not only interested in the validity, but also in the robustness of the method, as he obtains from experiments finite time series corrupted by noise. He is furthermore limited by the computer time necessary to run the computational algorithms.

Among the different quantitative measures of chaotic behaviors, one of the most widely used is the evaluation of the correlation dimension of the reconstructed attractor through the Grassberger–Procaccia algorithm [3]. This procedure indicates the minimum number of degrees of freedom necessary to account for the observed dynamics and characterizes the fractal nature of the attractor by a number which may be compared to those obtained from numerical simulations of models of the system under consideration. Among the whole spectrum of generalized dimensions [4], the correlation dimension D_2 is the easiest to compute because it needs fewer data points and does not require finite sample corrections [5].

While the Grassberger–Procaccia algorithm is rather easy to implement, numerous sources of systematic errors may often prevent the result from accurately reproducing the correlation dimension of the original attractor. These include for example the effect of random and digitizing noise [6], of low-pass filtering of the signal [7], of a too strong correlation between temporal neighbours on the trajectory [8].

We will try to show in the sequel that a strong nonuniformity of the attractor can also alter the determination of the correlation dimension, and how this effect may be substantially corrected by an adequate change of variable. This will be illustrated using experimental signals and numerical simulations of a CO_2 laser with modulated losses, comparing attractors reconstructed with the output intensity of the laser and with the logarithm of this intensity.

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2. Experimental system

The experimental signals presented hereafter come from a CO_2 waveguide laser with an electrooptic crystal and a ZnSe Brewster angle plate inserted inside the cavity of the laser [9]. A sinusoidal voltage is applied to the crystal at a frequency of 382.5 kHz, which results in modulation of the losses for the laser. The typical scenario encountered when increasing the modulation amplitude is a sequence of period doublings culminating in chaos [9]. The output intensity of the laser is measured with a HgCdTe detector and data acquisition is done with a LeCroy 9400 transient digitizer with a maximum sampling rate of 100 MHz, a storage capacity of 32000 samples and a resolution of eight bits.

The signals which may be seen on the first rows of figs. 1a and 2a have been recorded respectively at the end of the inverse cascade and further in the chaotic region. The intensity of the laser is often close to the zero intensity, so that a large number of points (15% and 45% in the case of figs. 1a and 2a respectively) in the time series have the same digitized value. When reconstructing the attractor using the method of time delays, small regions of the attractor contain most of the data points, unless very high embedding dimensions are used. This is a problem for accurately evaluating the correlation dimension, since the fractal structure is poorly resolved in those overpopulated parts of the attractor, due to the limited resolution of the digitizer. Similar signals with long periods of almost constant intensity may also be found in other chaotic lasers such as the laser with a saturable absorber [10] and the doped fiber laser with pump modulation [11]. Heavy low-pass filtering of the signal was used in ref. [12] to circumvent this problem. However, this introduces systematic errors as discussed by Badii et al. [7].

This feature of temporal signals in lasers can easily be explained by the fact that intracavity absorption and gain through stimulated emission are proportional to the intracavity radiation intensity I. This may be illustrated for example by the evolution equations for I and the inversion population D obtained from a single-mode, homogeneously broadened, two-level model for our laser with modulated losses [13]:

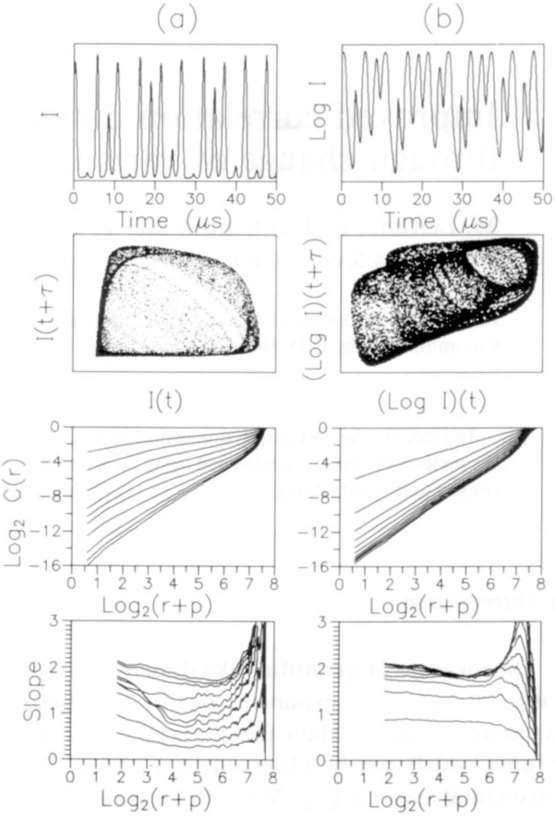

Fig. 1. Comparison of the attractors reconstructed with: (a) output of the HgCdTe detector, (b) output of the logarithmic amplifier. Row 1: simultaneous temporal sequences (625 samples). Row 2: phase portraits (20000 samples). $\tau=7\Delta t$, where $\Delta t=80$ ns is the sampling time. Row 3: log-log (base 2) plot of the correlation integral versus length scale $(r+p)$ for embedding dimensions 1 to 10 using τ as delay and 5000 data points. Row 4: slopes of the log-log plots of the correlation integral versus length scale for embedding dimensions 1 to 10.

$$\frac{dI}{dt}=2\kappa I(AD-1-m\sin\omega t)\,,$$

$$\frac{dD}{dt}=\gamma[1-D(1+I)]\,, \qquad (1)$$

where κ is the cavity damping rate, A the pump parameter, m and ω the modulation amplitude and frequency, γ the population inversion relaxation rate. The time derivative of I being proportional to I, the intensity seems to be frozen when it comes close to zero. It is then natural to use the logarithm of the intensity, whose time derivative is

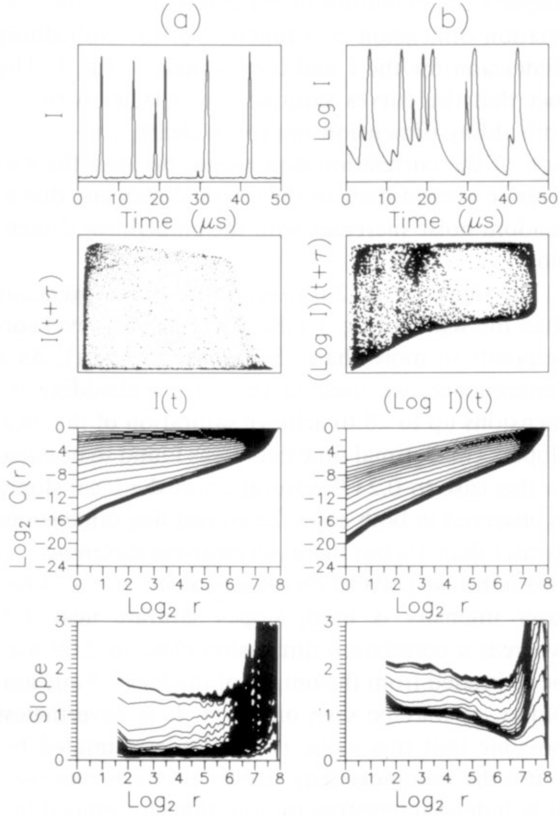

Fig. 2. Comparison of the attractors reconstructed with: (a) output of the HgCdTe detector, (b) output of the logarithmic amplifier. Row 1: simultaneous temporal sequences (625 samples). Row 2: Phase portraits (20000 samples). $\tau=8\Delta t$, where $\Delta t=80$ ns is the sampling time. Row 3: log-log (base 2) plot of the correlation integral versus length scale r for embedding dimensions 1 to 20 using τ as delay and 5000 data points. Row 4: slopes of the log-log plots of the correlation integral versus length scale for embedding dimensions 1 to 20.

$$\frac{d}{dt}\log I=2\kappa(AD-1-m\sin\omega t)\ ,$$

as the relevant dynamical variable. In other respects, Oppo et al. [14] remarked that the logarithm of the intensity appeared as a natural variable of their final equations in a paper in which they derived, by means of the center manifold theorem, two-dimensional equations for the CO_2 laser taking into account the coupling with rotational levels.

We have therefore inserted in our experimental setup a logarithmic amplifier between the detector and the digitizer. The output signal given by this amplifier is proportional to $\log(V_e+V_0)$, where V_e is the signal coming from the detector. The offset V_0 was chosen so that the zero intensity voltage of the detector was situated in the high slope region of the amplifier characteristics, in such a way as to discriminate the low intensity points without saturating too much the maxima. In each measurement the intensity signal coming directly from the detector and the output of the logarithmic amplifier were stored simultaneously to allow further comparison of the correlation integrals obtained in each case. The first rows of figs. 1b and 2b show the output signals of the logarithmic amplifier which were recorded simultaneously with the signals of figs. 1a and 2a respectively. These latter correspond to different modulation amplitudes and contain different amounts of low intensity periods. It is easy to see that the dynamics for the low intensity levels is well resolved. By inspecting the phase portraits on the second row of figs. 1 and 2, one can verify that the attractor reconstructed with the "logarithm" of the intensity is much more homogeneous and seems to be better reconstructed than the one obtained with the intensity.

3. Analysis of the experimental data

We present in this part the results obtained by analyzing these experimental data files with the Grassberger–Procaccia algorithm. Interpoint distances were computed using the maximum norm because, besides the fact that this speeds up significantly calculations, the systematic error induced by digitizing is more easily corrected than in the case of the Euclidean norm [6]. Indeed, because of the eight-bit resolution of the digitizer, we followed for the signal of fig. 1 a procedure suggested by Möller et al. [6], which consists in replacing r by $r+p$ in the log–log plots of the correlation integrals $C(r)$, where p is half the last significant bit of the digitizer. Such a correction was also used by Hübner et al. to compute dimensions and entropies in a NH_3 laser [15]. On the other hand, we did not use this correction in the case of the signal of fig. 2, because a noise level of the order of the last significant bit was estimated from the sudden slope increase in the log–log plots of the correlation integral. The digitizing error is substantially reduced in this case [6], and using the correction would overestimate the correlation dimension. It is possible that for the first data file the correction for the digitizing error, which assumes that the data are noise free, slightly overestimates the correlation dimension. However, our primary goal is to illustrate how attractor reconstruction depends on the variable used.

The third and fourth rows of fig. 1 display the plots of $\log C(r)$ versus $\log(r+p)$ for embedding dimensions up to 10 and of the local slopes of these curves versus $\log(r+p)$ for the signals of fig. 1. The time delay τ used in the reconstruction was chosen empirically as the one which yielded the widest scaling regions, and is approximately equal to $0.21T$, where T is the period of the modulation. Whereas no clear convergence of the local slopes may yet be seen in fig. 1a, where intensity was used, saturation is observed on fig. 1b for embedding dimensions greater than 6. The plateau in this case is only approximately 1.5 octaves wide, but it is rather difficult to get wider scaling regions with eight-bit resolution, if we exclude highly homogeneous attractors such as the Lorentz attractor [16]. The good convergence with the embedding dimension allows us to estimate the correlation dimension to be close to 2.05. Fig. 3 displays the evolution of the estimation of the correlation dimension as a function of the embedding dimension for the I and $\log I$ signals of fig. 1. The fact that the curves saturate much faster with the embedding dimension improves clearly the estimation of the correlation dimension, because the systematic overestimate of the fractal dimension due to random noise increases with the embedding dimension.

The signal of fig. 2 is much more inhomogeneous than the signal of fig. 1 (the zero intensity level corresponds to more than 75% of the samples). As a consequence, we used in this case embedding dimensions up to 20 to achieve saturation of the local slopes. The optimal time delay was found to be equal in this case to $0.24T$. Saturation of the local slopes is observed in both cases for embedding dimensions greater than 16, but there is a clear-cut difference between the two correlation dimension estimates. When laser intensity is used, slopes saturate near 1.5, whereas a correlation dimension close to 2.09 may be estimated from the output of the logarithmic amplifier, as may be seen on fig. 4. It is nevertheless possible that this value is still underestimated because the inhomogeneity is only partially removed. It is indeed interesting to note that, for embedding

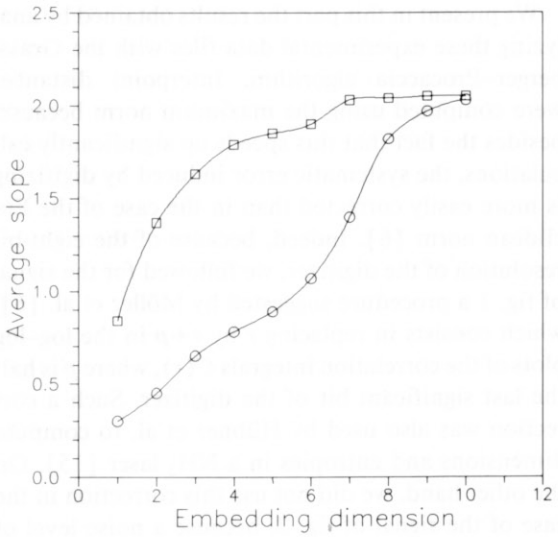

Fig. 3. Average slope in the "plateau region" versus embedding dimension for the data files of fig. 1a (circles) and fig. 1b (squares).

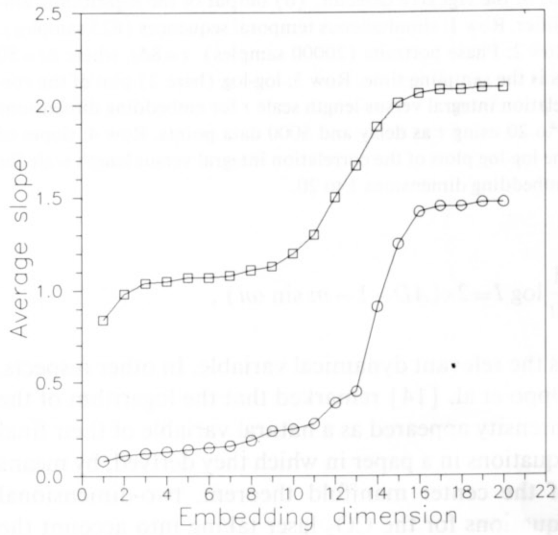

Fig. 4. Average slope in the "plateau region" versus embedding dimension for the data files of fig. 2a (circles) and fig. 2b (squares).

dimensions lower than 10, slopes of the correlation integrals are approximately equal to 0.15 in fig. 2a, indicating a very strong inhomogeneity, while in fig. 2b they cluster around 1.05. Thus, for low embedding dimensions, the reconstructed strange attractors look respectively like a point and a line.

For some other very inhomogeneous data sets, we were not able to find any clear scaling region in the log–log plots. We nevertheless systematically found that the local slopes using the logarithm were significantly higher than with the intensity and were always in the order of 2 for small distances.

4. Numerical simulations

We have also applied the same procedure to signals coming from numerical simulations, to check if similar effects could be observed even with higher precision, since in this case smaller length scales may be investigated. The set of equations (1) was numerically integrated with the Bulirsch–Stoer algorithm [17] and table 1 displays the parameters used. The signals were digitized to 32 bits. To compute the correlation integrals, we reconstructed the attractor using the method of delays, to ease the comparison with the experimental results.

To evaluate the fractal dimension with the best precision possible, since we are not limited in this case by random noise and digitizing errors, we used a maximum likelihood estimator of the correlation dimension derived by Takens [18]. The Takens estimator $D(r_0)$ of the correlation dimension for the length scales between 0 and r_0 is defined as

$$-\frac{1}{D(r_0)} = \langle \ln(r/r_0) \rangle ,$$

where the averaging is done on all the distances r between points of the attractor smaller than r_0. This

Table 1

Parameters	Value
κ	$6\times10^7\ s^{-1}$
A	1.1
ω	4×10^5 Hz
m	0.0246
γ	$2.5\times10^5\ s^{-1}$

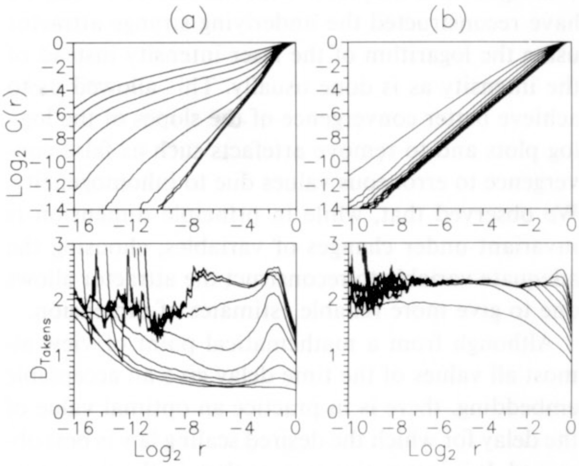

Fig. 5. Analysis of the files obtained from numerical simulations using: (a) intensity, (b) logarithm of the intensity. Row 1: log-log plot of the correlation integral versus r for embedding dimensions 2 to 10. Row 2: Takens estimator of the correlation dimension $D_{\mathrm{Takens}}(r)$ versus $\log_2 r$ for embedding dimensions 2 to 10.

method, which has the advantage of being non-parametric, is more difficult to use with digitized data of relatively low precision since there is in this case a large incertainty in the logarithms of the distances.

Correlation integrals and the Takens estimator for the numerical simulations may be seen in fig. 5. The calculations were made with 5000 points, and a delay time equal to $0.24T$. The benefit in using the logarithm of the intensity to reconstruct the attractor is even more clearly demonstrated than for the experimental data, probably because we may use here the true logarithm. For embedding dimensions higher than 3, we observe a clear convergence to a value of 2.29 over nearly three octaves for the logarithm, whereas no convergence may be seen with the curve for the intensity. To have independent information on the fractal dimension of the attractor, we computed the Lyapunov dimension of the attractor [19] which is known to be an upper bound of the correlation dimension [20]. This yielded a value of 2.304, which is very close to the correlation dimension found by the Takens estimator.

5. Conclusion

To evaluate the correlation dimension of signals coming from a CO_2 laser with modulated losses, we have reconstructed the underlying strange attractor using the logarithm of the laser intensity instead of the intensity as is done usually. This allowed us to achieve better convergence of the slopes of the log–log plots and to remove artefacts such as false convergence to erroneous values due to inhomogeneity. We observed that, while in principle dimension is invariant under changes of variables, choosing the adequate variable to reconstruct the attractor allows one to give more reliable estimates of dimension.

Although from a mathematical point of view almost all values of the time delay give an acceptable embedding, there is in practice an optimal value of the delay for which the desired scaling law is best observed. It is interesting to note that, in the same way, while in principle reconstruction of the attractor may be done using any function of the intrinsic dynamical variables, we may conjecture from the results of this study that there should be an optimal function for the reconstruction of the attractor. In the case of our laser, the logarithm of the intensity happens to be a more natural and more efficient variable for quantitative characterization than the intensity itself. The method may obviously be used in the study of other chaotic systems with long sequences of almost constant signal [10,11].

We believe that using the logarithm of the output intensity of a laser should also prove useful for other techniques of characterizing deterministic chaos, such as symbolic dynamics [21], extraction of unstable periodic orbits [22] and template analysis [23], since it provides more details on the regions of the reconstructed phase space which are squeezed.

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