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Symbolic Dynamics in a Passive Q-Switching Laser.

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Abstract. – Passive Q-switching CO_2 lasers may exhibit chaotic behaviour. This behaviour is studied here using symbolic dynamics. Both numerical simulations and experimental data are coded with hypersymbols. This allows us to evaluate the metric complexity of the chaotic behaviour.

Under appropriate conditions, CO_2 lasers containing an intracavity saturable absorber (LSA) present a self-pulsing behaviour called passive Q-switching (PQS) [1]. This term covers a family of regimes resulting from the interaction of a Hopf and a homoclinic bifurcation [2]. When a suitable control parameter is used, the typical evolution is a sequence of periodic and stochastic regimes, such as observed in the Belouzov-Zhabotinski chemical reaction [3]. A phase space analysis of the stochastic trajectories, together with the construction of 1D maps, reveals their chaotic nature and underlying mechanisms [2].

The next step in the characterization of such a behaviour is quantitative. This may consist in the calculation of the dimensions and Kolmogorov entropy associated with the chaotic regimes. These calculations are typically based on a phase space reconstruction of the chaotic attractor [4]. In the case of homoclinic chaos, symbolic dynamics (SD) offers another approach since there exists a natural coding of the signal [2, 3].

SD is a powerful tool for the study of nonlinear dynamical systems [5]. It links local and global structures, connecting, for example, dynamical variables to thermodynamic ones. For the experimentalist, two complementary approaches may be distinguished: i) one considers the signal as «stochastic», *i.e.* as a Markov chain, and measures its degree of correlation through its Markov order [6], ii) one evaluates the «order» of the signal with the depth of the hierarchy of the symbols needed to represent the data. This is the quantitative characterization of chaos defined by Badii [7].

In this paper, we present a preliminary symbolic analysis of the chaotic regimes of LSA. We first analyse the signal using a simple coding. This reveals the presence of hypersymbols which allows us to evaluate the complexity of the signal. The robustness of the results of numerical simulations has been checked with special care. Rules for the application of these methods to experimental data have been derived. They have been used on signals from our experiments on the CO_2 laser with CH_3I as a saturable absorber.

Results presented here have been collected in a chaotic region called $C^{(2)}$ [8] where the laser emits a succession of large and small pulses, as shown on the experimental record of fig. 1. The coding simply consists of transforming the temporal sequence into a binary chain associated with the height of the peaks: the large ones are coded as a «L» and the small ones as a «S». For instance, the sequence of fig. 1 will be coded as:

SLSSLSSLSLSSLSSLSSLSLSLSSLSSLSSLSSLLSLSSLSSLS

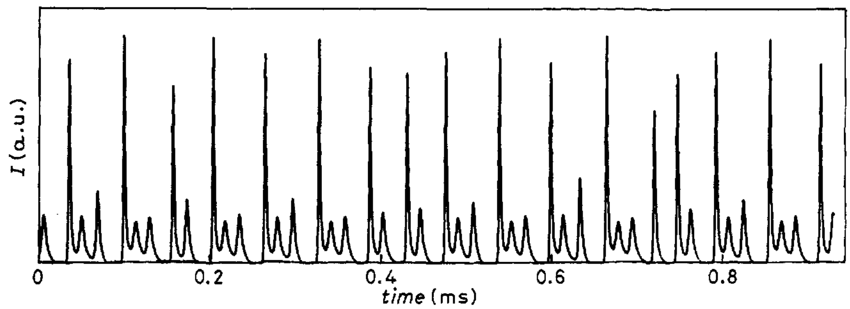

Fig. 1. – Example of the $C^{(2)}$ regime lying at the edge of the $P^{(2)}$ periodic regime. This is a result from a numerical simulation performed with the parameters of ref. [2] and $A = 1.888$.

A well-established simple model accurately reproduces this behaviour [2, 9] and allows us to check the robustness of the methods when applied to limited sets of data, such as those provided by the experiments. Once applicability criteria of these methods have been derived, they can be used to treat experimental data.

Numerical results. – The model used for numerical simulations has been extensively discussed [2, 9]. The amplifier is a three-level system and the absorber is a two-level one. The parameter values used in the present numerical simulations are the same as in [2]. In a first step, the chain is considered as a Markov process of high order. Let us recall that a sequence of n symbols $X_1 X_2 X_3 X_4 \ldots X_n$ is similar to a Markov process of k-th order if the conditional probability of observing X_L after a succession of $L-1$ symbols obeys the relation

$$P(X_L | X_1 \ldots X_{L-1}) = P(X_L | X_{L-k} \ldots X_{L-1}) .$$

The Markov order may be seen as the memory of the signal, *i.e.* the number of symbols influencing the L-th one. A simple way to evaluate the Markov order is to perform a statistic test such as [6]

$$\sigma(x)^2 = \frac{1}{n^L} \sum_{i=1}^{n^L} \frac{[P_i - P_i(x)]^2}{P_i} ,$$

where $P_i = P(X_1 X_2 \ldots X_L)$ is the existence probability of the word $X_1 X_2 \ldots X_L$ and $P_i(x)$ is the probability to observe this sequence in a stochastic process of order x. $\sigma(x)^2$ becomes equal to zero when x is larger than the Markov order k.

A plot of the evolution of $\sigma(x)$ *vs.* x has been done for various values of n and L. In fig. 2*a*), we display such a plot (solid line) for $n = 10^6$ and $L = 10$. Similar results have been

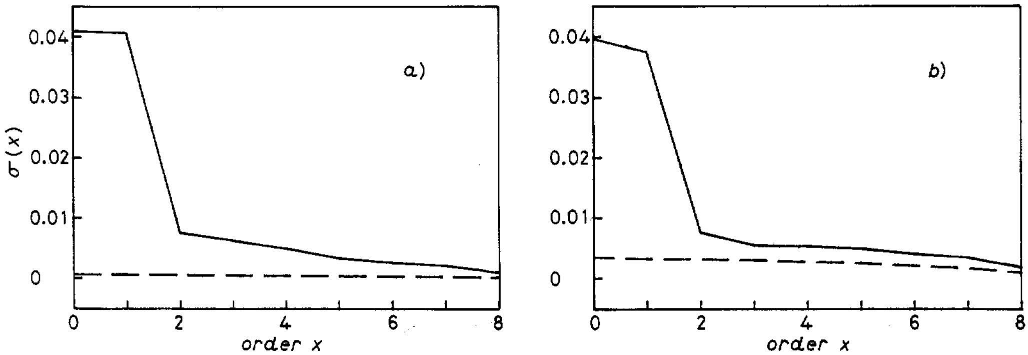

Fig. 2. – Determination of the Markov order k through the statistic test $\sigma(x)$. The full lines correspond to the coding with binary symbols, while the dashed ones correspond to the coding with hypersymbols, for *a*) numerical simulations in the conditions of fig. 1 and *b*) experimental data in similar conditions.

obtained for smaller L values, *e.g.* 10^4, 10^5, ... The sudden change in the slope for $x = 2$ suggests that the chaotic sequence may be considered as a Markov process of order two. This indicates that there should exist hypersymbols that better represent the dynamics of the signal.

The search for hypersymbols is made through the construction of a logic tree, as proposed by Badii [7]. Two criteria are considered for the location of a word in the tree: i) its existence and ii) its periodicity. A word is p-periodic if the chain of length p made by the repetition of the word exists. If a word exists and it is not periodic, it is called a «phantom» and does not appear in the tree. A periodic word is «primitive» if it cannot be decomposed into a concatenation of primitives. So a primitive is either a word of one symbol, or the concatenation of a phantom and another primitive. All primitives appear at the level 1 of the tree.

It is clear that the optimum situation corresponds to infinite-length chains with a very large periodicity. For instance, Badii worked on chains of length 10^8 with a periodicity of about 20. In our case, this does not correspond to any realistic experimental situation since it is very difficult to sample experimental chains of more than 10^4 characters. Numerical simulations have been performed on similar data sets producing chains of up to 10^6 characters to check the results. It appeared that a periodicity large compared to the log of the length of the chains leads to erroneous results. Typically, the periodicity must be reduced to 6 for 10^6 points and 5 for 10^4. The low value of these optimal periodicities seems to be due to the existence of symbols with very low probability.

If these rules are respected, we find in the conditions as in fig. 2 that S is a phantom and that there exist three primitives: L, SL and SSL. An obvious consequence of this result is that the signal can be coded by three hypersymbols $0 \equiv \mathrm{L}$, $1 \equiv \mathrm{SL}$ and $2 \equiv \mathrm{SSL}$. This result is not surprising since it corresponds to the intuitive code already used for the analysis of 1D maps of the LSA [2].

An evaluation of the Markov order of the chain coded by hypersymbols gives $k = 0$ (dashed line of fig. 2*a*)). So the coding in 0, 1, and 2 appears to be the most elaborated one, it contains all the memory of the signal and is particularly appropriate for the construction of the logic tree. The first three levels obtained by this coding are given in fig. 3.

The code with the hypersymbols again reduces the maximum allowed periodicity for the building of the tree. We have checked that we had to limit the maximum periodicity to 5 for 10^6 hypersymbols and 4 for 10^4 ones. So we were able to build the tree up to level 3 since the primitives here are one character long.

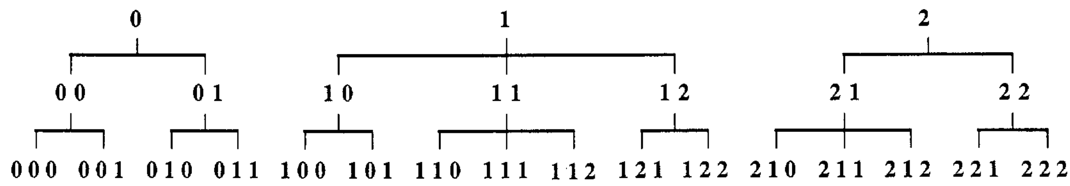

Fig. 3. – First three levels of the logic tree for numerical simulations with parameters of fig. 1 and coding with the hypersymbols 0, 1 and 2.

Starting from the second level, some grammatical rules limit the number of possible combinations of each level. For example, at level 2, the combinations 02 and 20 do not appear in the tree; they are not periodic. At level 3, a larger number of words disappear. In fact, some of these words are not only nonperiodic, but are even forbidden. For example, 120 denotes the grammatical rule: «the word 12 is never followed by a 0». These rules appear from the third on and occur predominantly from the fourth level of the tree on. The complexity of these rules can be measured using the «metric complexity» C_1: if the knowledge of level m permits to predict exactly the combinations present at level $m+1$, that means that no grammatical rule exists and the system may be considered as «simple» and so of zero complexity. The maximum complexity corresponds to a situation where the predictions systematically fail. Badii proposed a definition of the metrix complexity [7]:

$$C_1 = \lim_{m\to\infty} \lim_{p\to\infty} \frac{1}{\ln N_0(m,p)} \sum_{j}^{N(m,p)} P_i \ln \frac{P_i}{P_{0i}},$$

where $N(m, p)$ is the number of words of the m-th level after consideration of all p-periodic orbits, and $N_0(m, p)$ is the number of orbits predicted at level m from the knowledge of all periodic orbits up to length $p-1$, P_{0i} is the predicted probability, *i.e.* the product of the probability of the two words concatenated to form the final word.

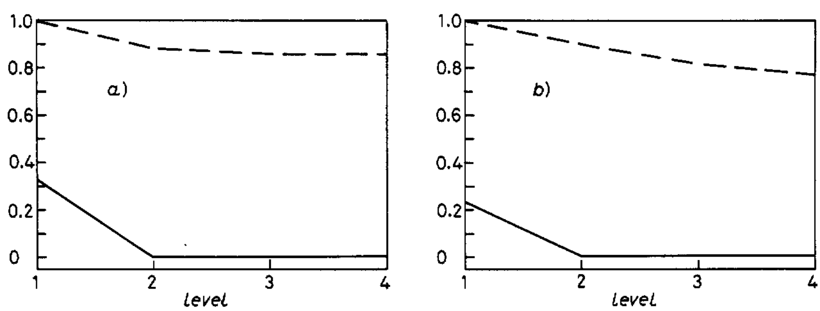

Fig. 4. – Evolution of the complexity as the logic tree level of computation is increased (full lines). The dashed lines represent the total probability of the level, *i.e.* its filling. *a*) Numerical simulations with parameters of fig. 1 and *b*) experimental data as in fig. 2*b*).

Figure 4*a*) displays the plot of the value of C_1 as a function of the level m is the same conditions as those of fig. 2. The value of the total probability of the levels is also shown, indicating to what degree that level is filled. From these, C_1 is evaluated as $5 \cdot 10^{-3} \pm 1 \cdot 10^{-3}$. The computation of C_1 for a sample of 10^6 points gives similar results, showing that the convergence of C_1 as a function of the level is obtained after a few steps.

This quantity can be used to follow the evolution of the dynamics in the chaotic window $C^{(2)}$. Figure 5 shows a plot of the complexity as a function of the pump parameter. The regime evolves from $P^{(2)}$ to $P^{(3)}$ from left to right. We verified that the sudden change in the slope of the curve corresponds approximatively to the appearance of $P^{(3)}$-like pulses in the chaotic signal.

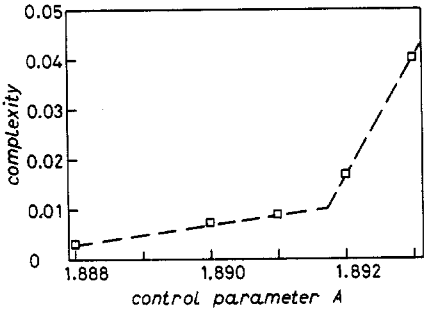

Fig. 5. – Evolution of the complexity in the $C^{(2)}$ chaotic window. The dashed line does not result from a fit and has been plotted only to facilitate the reading of the figure.

Finally, it appears that in these numerical simulations, SD allows us to compute a quantity that characterizes the chaos exhibited by this system. The metric complexity has been evaluated with a resolution sufficient to detect changes in the nature of the chaotic signal. Limits regarding periodicity and sequence length have been determined, beyond which accurate results cannot be obtained. These limits are respected in the following analysis of experimental data.

Experiments. – The chaotic signals to be analysed were obtained from the $CH_3I + CO_2$ LSA described elsewhere [10]. Experiments have been performed in conditions corresponding to the numerical studies presented above, *i.e.* with a laser operating in the $C^{(2)}$ chaotic regime close to the limit of the $P^{(2)}$ regime. The signal which is proportional to the laser output intensity is stored in a digital oscilloscope and then transferred to a computer which converts it into symbols and/or hypersymbols.

Plots of the Markov order for both the binary code and the hypersymbols are given in fig. 2*b*) for a sample of about $3 \cdot 10^4$ binary codes (about 10^4 hypersymbols). Its remarkable similarity with fig. 2*a*) allows us to draw the same conclusions concerning the correlation time of the observed chaotic signals: the binary coded chains have a two-character «memory». This memory is accounted for in the hypersymbol coding.

Because of the limited amount of experimental data, the tree was restricted to period ≤ 4, as discussed in the preceding section. Figure 4*b*) shows the plot of the evolution of the complexity *vs.* m for a regime close to the $P^{(2)}$ periodic one, *i.e.* in the same conditions as fig. 2. The final value of $6 \cdot 10^{-3}$ is in very good agreement with the theoretical one. The present version of our apparatus did not allow us to sample with sufficient accuracy chains for different control parameter values, so we could not verify the results of fig. 5. However, a second series of experiments leads to a complexity of $2.5 \cdot 10^{-2}$ for a chaotic signal recorded at the edge of the appearance of the $P^{(3)}$-like pulses, *i.e.* in conditions similar to $A = 1.892$ in the model. This is remarkably close to the numerical results.

Finally, the good agreement between numerical and experimental results shows again that the simple model used here provides good hints of the behaviour of the LSA [2].

Discussion. – The main goal of these studies was to devise new quantitative measurements of chaos applicable to experimental data sets. In contrast with numerical simulations, reliable data cannot be stored for very long times owing to limitations in both the stability of the laser and the memory size of fast transient recorders. Data reduction implied by symbolic dynamics forces us to treat time series much longer than those required by other methods of quantitative chaos analysis. This drawback seems to put SD at a disadvantage. However, contrary to other methods, SD is not sensitive to detection and digitalization noise. Moreover, a careful tuning of the parameters of the computation allowed us to derive reliable evaluations of the complexity C_1 of the experimental signals. The fact that C_1 is significantly different from 0 is characteristic of a chaotic system that is topologically not simple and provides a quantitative measure of chaos in our laser.

This first approach of the application of symbolic dynamics to the LSA helped us to characterize the chaos which was observed in this system. The next step would be to study how this complexity evolves as some control parameter drives the laser through the chaotic domain and possibly to extend the calculations to the generalized complexities [7].

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